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MODIFICATION OF THE METHOD OF DISCRETE CONTINUATION BY PARAMETERS

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We consider a system of nonlinear equations

$$F_i(x_1, x_2, \dots, x_n, p) = 0 \quad (i = \overline{1, n}), \tag{1}$$

where x_i $(i = \overline{1, n})$ are arguments and p is the solution parameter. Nonlinear problems of mechanics can often be reduced to systems of this kind. One such elementary problem is the problem of axisymmetric buckling of an isotropic circular plate acted upon by radial forces N_0 distributed uniformly on the contour and by a transverse load q:

$$\begin{gathered} r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r^2N_r)\right] + \frac{Eh}{2}\left(\frac{dw}{dr}\right)^2 = 0, \\ \frac{D}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} - \frac{1}{r}\frac{d}{dr}\left(rN_r\frac{dw}{dr}\right) = q \quad (0 \leqslant r \leqslant R), \\ dw/dr = Q_r = u_r = 0 \text{ for } r = 0,\ w = M_r = 0,\ N_r = -N_0 \text{ for } r = R. \end{gathered} \tag{2}$$

Here, u_r and w are the radial displacement and the deflection; N_r, M_r, and Q_r are the specific radial force, the bending moment, and the shearing force; E and D are Young's modulus and the cylindrical rigidity of the plate; R and h are the plate's radius and thickness, respectively.

We propose to construct the loading trajectory of a mechanical object whose behavior is described by system (1):

$$x_i = x_i(p)\ (i = \overline{1, n}).$$

The method of continuous parameter continuation is convenient for solving this problem. With this method one constructs the loading trajectory at all points that are regular in the Poincaré sense, including the limiting points of the trajectory. The idea of this method was first advanced in [1]. Its detailed elaboration, which considers the equivalence of solution variables, was given in [2]. However, this method has a shortcoming: In the course of numeric construction of the loading trajectory, an uneliminable error accumulates in the solution. After several steps in the continuous continuation method, one has to adjust its solution. This adjustment is done by an algorithm that relies on the techniques of the method of discrete continuation in the parameter, which also implements the concept of equivalence of parameters [2]. On this basis, one can adjust the solution at regular and limiting points of the trajectory. Without reviewing the various methods of continuous and discrete continuation (such a review can be found in [2, pp. 12-23, 176-196]), we will examine

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 95-99, September-October, 1990. Original article submitted May 16, 1988; revision submitted April 6, 1989.

 0021-8944/90/3105-0754$12.50

three adjustment variants. The first of these is well known. It is taken as the basis for the other two algorithms (which have never been published).

We assume that the parameter p is a variable on a par with x_i ($i = \overline{1, n}$): $x_{n+1} = p$. Suppose that, by the continuous continuation method, a solution x_i ($i = \overline{1, n+1}$) has been obtained and must be adjusted. According to Newton's method, we write at this point of the loading trajectory the algebraic system for errors of (1):

$$\sum_{j=1}^{n+1} F_{i,j}\Delta x_j = -F_i \quad (i=\overline{1,n}), \quad F_{i,j} = \frac{\partial F_i}{\partial x_j}, \quad x_j = x_j + \Delta x_j \quad (j=\overline{1,n+1}). \tag{3}$$

The system consists of n linear equations for n + 1 unknown quantities Δx_j. Its definition must be completed so that Δx_j ($j = \overline{1, n+1}$) will be defined at any regular point on the trajectory with the maximum possible conditionedness. The following procedure was suggested for this purpose in [2].

With the method of continuous continuation applied at any point of the loading trajectory, we can define a unit vector which is tangential at a given point to the loading trajectory:

$$\Phi = (\varphi_1\varphi_2 \ldots \varphi_{n+1})^T, \ \dot{x}_i = \varphi_i \ (i=\overline{1,n+1}).$$

Here, $\dot{x}_i$ ($i = \overline{1, n+1}$) is the derivative of the variable x_i with respect to an arc λ of the loading trajectory. It can be demonstrated (see [2]) that if we try to find the increment Δx_j ($j = \overline{1, n+1}$) in the direction perpendicular to the vector Φ, the process of search for the increment will be best conditioned at each regular point of the loading trajectory. The best conditionedness at these points is equal to the maximum conditionedness of one of the following matrices:

$$\mathbf{J}_k = \begin{pmatrix} F_{1,1} & \ldots & F_{1,k-1} & F_{1,k+1} & \ldots & F_{1,n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ F_{n,1} & \ldots & F_{n,k-1} & F_{n,k+1} & \ldots & F_{n,n+1} \end{pmatrix} \quad (k=\overline{1,n+1}).$$

The process leads to this system for finding the increments:

$$\begin{pmatrix} F_{1,1} & \ldots & F_{1,n} & F_{1,n+1} \\ \cdot & \cdot & \cdot & \cdot \\ F_{n,1} & \ldots & F_{n,n} & F_{n,n+1} \\ \varphi_1 & \ldots & \varphi_n & \varphi_{n+1} \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \cdots \\ \Delta x_n \\ \Delta x_{n+1} \end{pmatrix} = -\begin{pmatrix} F_1 \\ \cdots \\ F_n \\ 0 \end{pmatrix}. \tag{4}$$

At large n, the process for each iteration takes an amount of time proportional to $2n^3$ if the solution of (4) is found by the orthogonalization method (as suggested in [2]), or to $2n^3/3$ if Gaussian-type methods are employed. The memory capacity required in both cases is proportional to n^2.

The second alternative of specification of the loading trajectory parameters should be applied when the Jacobian matrix of (1) is symmetric:

$$\mathbf{J}_{n+1} = \begin{pmatrix} F_{1,1} & \ldots & F_{1,n} \\ \cdot & \cdot & \cdot \\ F_{n,1} & \ldots & F_{n,n} \end{pmatrix}, \quad F_{i,j} = F_{j,i} \quad (i, j = \overline{1,n}).$$

In this case, (3) is supplemented with the equation $F_{1,n+1}\Delta x_1 + \ldots + F_{n,n+1}\Delta x_n + \varepsilon\Delta x_{n+1} = 0$. This equation means that the direction in which we seek the increments Δx_i ($i = \overline{1, n+1}$) is perpendicular to the vector $\Phi^* = (F_{1,n+1} \ldots F_{n,n+1}\varepsilon)^T$ (ε is the indefinite parameter to be determined later). The resulting equations system for determination of Δx_i ($i = \overline{1, n+1}$) has a symmetric matrix

$$\begin{pmatrix} F_{1,1} & \ldots & F_{1,n} & F_{1,n+1} \\ \cdot & \cdot & \cdot & \cdot \\ F_{n,1} & \ldots & F_{n,n} & F_{n,n+1} \\ F_{1,n+1} & \ldots & F_{n,n+1} & \varepsilon \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \cdots \\ \Delta x_n \\ \Delta x_{n+1} \end{pmatrix} = -\begin{pmatrix} F_1 \\ \cdots \\ F_n \\ 0 \end{pmatrix},$$

which is less conditioned the greater is the angle between the vectors Φ and Φ^*. Therefore, we should try to find ε in such a way as to minimize the angle between the vectors Φ and Φ^*. On the basis of this condition, we write

$$\alpha = \alpha_{\min} \text{ for } \varepsilon = \varphi_{n+1} \sum_{i=1}^{n} F_{i,n+1}^2 \Big/ \sum_{i=1}^{n} F_{i,n+1}\varphi_i.$$

This variant of adjustment requires for symmetric matrices J_{n+1} a memory capacity proportional to $(n^2 + n)/2$; the time of realization of an iteration with a Gaussian method is proportional to $n^3/3$.

In the third variant of solution adjustment, the new approximation of (3) is sought in the direction defined by the gradients of the functions F_i ($i = \overline{1, n}$):

$$\Delta X = \sum_{i=1}^{n+1} \Delta x_i e_i = \sum_{j=1}^{n} t_j \operatorname{grad} F_j = \sum_{j=1}^{n} \sum_{i=1}^{n+1} t_j F_{j,i} e_i$$

[t_j ($j = \overline{1, n}$) are the parameters that specify the direction where the adjustment of the solution should be sought]. Hence,

$$\Delta x_i = \sum_{j=1}^{n} t_j F_{j,i}. \tag{5}$$

Substituting (5) into (3), we obtain an equations system for t_j ($j = \overline{1, n}$):

$$\mathbf{HT} = -\mathbf{F}, \tag{6}$$

where

$$\mathbf{H} = \mathbf{J}^*\mathbf{J}^{*T};\ \mathbf{T} = (t_1 t_2 \ldots\ldots t_n)^T;\ \mathbf{F} = (F_1 F_2 \ldots\ldots F_n)^T;$$

$$\mathbf{J}^* = \begin{pmatrix} F_{1,1} & \ldots & F_{1,n} & F_{1,n+1} \\ \cdot & \cdot & \cdot & \cdot \\ F_{n,1} & \ldots & F_{n,n} & F_{n,n+1} \end{pmatrix}.$$

Solving (6), we find t_j ($j = \overline{1, n}$). On this basis, from (5) we obtain ($i = \overline{1, n+1}$).

The matrix $\mathbf{H}$ is always symmetric and of order n. Therefore, for solution of (6) we need a memory capacity of $(n^2 + n)/2$ and a time proportional to $n^3/3$ if a Gaussian method is used. However, we should also count the capacity for memorizing the matrix $\mathbf{J}^*$ and the time of formation of the matrix $\mathbf{H}$. Thus, the total time for implementation of an iteration will be proportional to $4n^3/3$, and the memory to n^2 (if the matrix $\mathbf{J}_{n+1}$ was initially symmetric). If $\mathbf{J}_{n+1}$ was asymmetric, the realization time of an iteration remains the same, but the required memory capacity grows to $(3n^2 + n)/2$.

The iteration process realized with the aid of (6) has the maximum possible conditionedness at any step in the neighborhood of a regular point or the limiting point of the loading trajectory, because the matrix $\mathbf{H}$ is not singular at these points. Its determinant is

$$\det(\mathbf{H}) = (-1)^n \sum_{k=1}^{n+1} [\det(\mathbf{J}_k)]^2.$$

The matrices $\mathbf{J}_k$ ($k = \overline{1, n+1}$) degenerate simultaneously only at the branching point of the trajectory. With this conditionedness of the process, the laboriousness of the iterations can be reduced. Without any significant deterioration of convergence, we can form the matrix $\mathbf{H}$, invert it only at the first iteration, and subsequently adjust the solution with the aid of the inverse matrix. Now for an iteration we will need the time on the order of $4n^2$. The number of iterations with the continuous continuation method taken as the initial approximation has been estimated to be increased by one or two.

With these techniques the accuracy of solutions of a nonlinear equations system by the method of continuous continuation can be greatly improved, the laboriousness can be reduced by increasing the integration step on the trajectory, and the stability of the solution can be enhanced in the neighborhood of the branching points of the loading trajectory. For example, in solving the axisymmetric problem of combined loading of a circular plate (2) on the basis of an exact solution in a Way power series [3], these techniques made it possible to traverse all branches of the loading trajectory. The trajectory is depicted in Figs. 1-3.

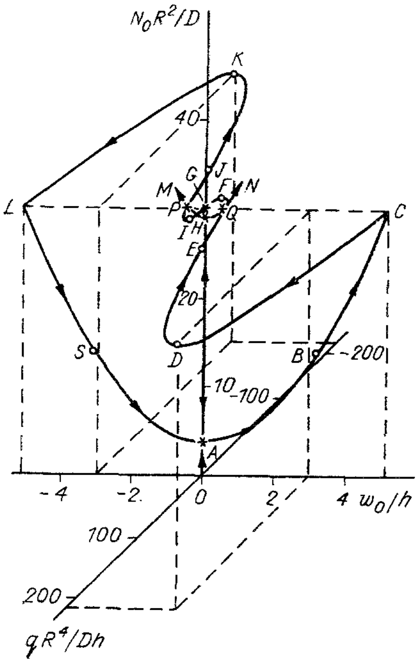

Fig. 1

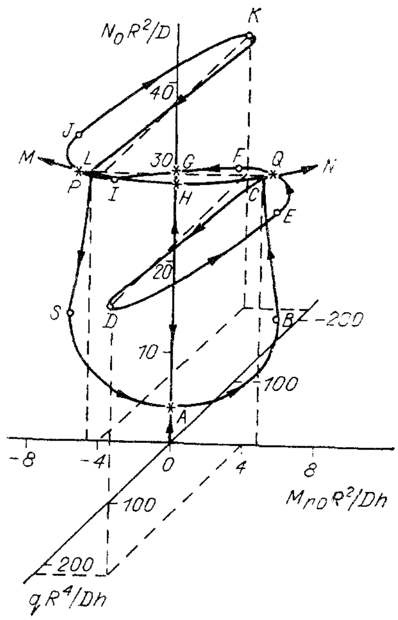

Fig. 2

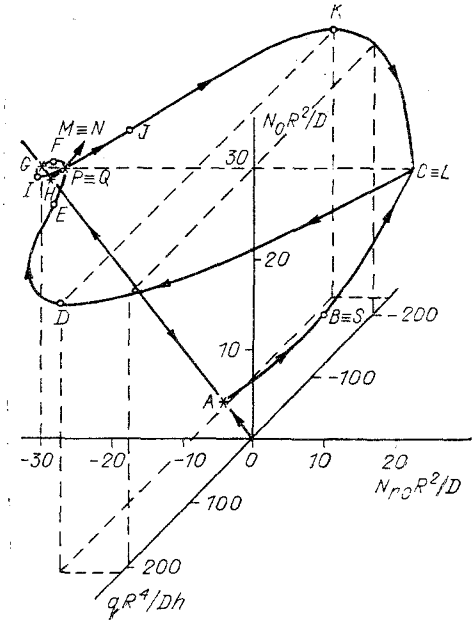

Fig. 3

Here, w_0, M_{r0}, and N_{r0} are the deflection, the bending moment, and the tensile force at the plate center. Initially, the plate is loaded by a uniform contour load N_0. This force causes the plate to lose stability in the first axisymmetric form (point A). It is brought along the bifurcation branch ABC to the transcritical region, where for N_0 the value of $30D/R^2$ (point C) is formed. After that, a uniform transverse load q is applied to the plate on the convex side. It buckles along the branch CDEQFGIPJKL, like a low-angle spherical dome. The loading trajectory includes limiting points B, D, E, F, I, J, K, and S and bifurcation points A, H, G, P, and Q. Points P and Q are intersections of the plate's loading trajectory with the bifurcation branch, corresponding to the second axisymmetric form of stability loss.

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OSCULATION OF TWO NONLINEARLY ELASTIC BODIES

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UDC 539.3

The problem of osculation of two solids S_1 and S_2 similar in shape to a half-plane and made of a nonlinearly elastic harmonic-type material is investigated [1]. The contact area is assumed to be free of friction. An exact solution is obtained.

1. We consider physical regions S_1 and S_2 with boundaries of close-to-linear shape. After deformation, they come into contact along the common portion L of their respective boundaries L_1 and L_2. The contact of the bodies is accomplished by external forces, whose principal vector is $P_0 = X + iY$ (P_0 is a known constant). The contact region L is assumed to consist of a finite number of segments of the real axis ox: $L = [a_1 b_1] + \ldots + [a_n b_n]$. Suppose that S_1 and S_2 occupy the lower and upper half-planes of the plane of the variable $z = x + iy$ [2]. Quantities referring to S_1 and S_2 will be identified by subscripts 1 and 2, respectively. Stresses and rotations are absent for these bodies at infinity.

The boundary conditions of the problem are [3]

$$v_1^- - v_2^+ = f(\overset{*}{x}),\ T_1(x) = T_2(x) = 0,\ N_1(x) = N_2(x) = N(x) \text{ on } L, \tag{1.1}$$

On the remaining portions of the boundaries that are free of external actions

$$N = 0,\ T = 0. \tag{1.2}$$

Here N and T are the normal and the tangential stresses; v is the normal elastic displacement; $f(\overset{*}{x}) = f_2(\overset{*}{x}) - f_1(\overset{*}{x})$ is a function specified on the deformed contact line; f_1 and f_2 characterize the configuration of the compressed bodies after deformation. It will be recalled that $\overset{*}{x} = x + u$, $u = u(x)$ is the horizontal elastic displacement of the points of the line L. We will assume that $f'(\overset{*}{x}) \in H(L)$.

The solution makes use of a complex representation of the fields of elastic elements for a half-plane in terms of two functions $\varphi(z)$ and $\psi(z)$ of the complex argument $z = x + \iota y$ which are analytic in that half-plane [4]:

$$X_x + Y_y + 4\mu = \frac{(\lambda + 2\mu)\, q\Omega(q)}{\sqrt{J}}, \quad Y_y - X_x - 2\iota X_y = -\frac{4(\lambda + 2\mu)}{\sqrt{J}} \frac{\Omega(q)}{q} \frac{\partial z^*}{\partial z} \frac{\partial z^*}{\partial \bar{z}}; \tag{1.3}$$

$$u + \iota v = \frac{\mu}{\lambda + 2\mu} \int \varphi'^2(z)\, dz + \frac{\lambda + \mu}{\lambda + 2\mu} \left[\frac{\varphi(z)}{\overline{\varphi'(z)}} + \overline{\psi(z)} \right] - z; \tag{1.4}$$

$$\frac{\partial z^*}{\partial z} = \frac{\mu}{\lambda + 2\mu} \varphi'^2(z) + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\varphi'(z)}{\overline{\varphi'(z)}}, \quad \frac{\partial z^*}{\partial \bar{z}} = -\frac{\lambda + \mu}{\lambda + 2\mu} \left[\frac{\varphi(z)\, \overline{\varphi''(z)}}{\overline{\varphi'^2(z)}} - \overline{\psi'(z)} \right], \tag{1.5}$$

where

$$z^* = z + u + \iota v; \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \iota \frac{\partial}{\partial y}\right); \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \iota \frac{\partial}{\partial y}\right);$$

Tbilisi. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 99-104, September-October, 1990. Original article submitted April 5, 1989.

